Optimal Patent Length in A Model of Mortality Increasing with Age

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Abstract

Macroeconomic models using representative individuals estimate optimal patent lengths that are much longer than current actual patent length. In the forever young overlapping generation model under the setting of fixed mortality, extending patents expands the inter-generational wealth gap, and the estimated optimal patent length is shorter, but still longer than the current actual patent length. When we consider the phenomenon that mortality increases with age, the mortality rate of older people who benefit from patent extension is higher, resulting in the optimal patent length being shorter. Although it is still longer than the current actual patent, it is already very closer.

Keywords: Social Welfare, Overlapping Generations, Patent Length, Research and Development (R&D), Intergenerational Wealth Distributions

JEL Classification: O30, O40

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1. Introduction

The main purpose of the government to provide patent protection is to encourage firms to engage in research and development, so that successful innovative firms retain the monopoly right to produce innovative goods for a period of time, and then enable successful innovative firms to earn monopoly profit during the patent protection period. With the protection of patents, firms are willing to bear the cost of R&D and innovation in advance. However, the monopoly pricing of innovative firms makes the price of goods too high, causing social deadweight loss. Therefore, there is an optimal length of patent protection. If the protection period is too short, the monopoly profits obtained by successful innovative firms are not enough to pay the R&D expenses that need to be paid in advance, and firms are not willing to engage in R&D, thereby reducing the level of technological innovation. On the contrary, if the protection period is too long, consumers buy goods at higher prices for a long time, resulting in a large amount of deadweight loss of social welfare. This is what Reinganum (1989) points out: patent protection has extremely important trade-offs. Stimulating R&D brings dynamic gains, while reducing competition brings static losses.

Since Romer (1990) proposes the endogenous growth model considering research and development (R&D), many literatures begin to analyze the patent length issue based on Romer's technological progress model of diversified intermediate goods. Kwan and Lai (2003) is the first paper to comprehensively and quantitatively analyze optimal patent length using the Romer model. The optimal patent length depends on the marginal loss of reducing competition equaling the marginal gain of stimulating R&D. Kwan and Lai (2003) also conduct a numerical analysis of the length of US patents. The conclusion of the analysis is that the patent term in the United States is not optimal, and the welfare loss caused by over-protection is relatively lower than that of under-protection. It is suggested that the government should extend the patent term. However, there are also many documents that do not support the extension of patent protection period. Merges and Nelson (1994) find that many countries have imperfect patent systems and technological progress is still rapid. Klinger and Lederman (2006) point out that the lack of threat of imitators reduce the motivation of successful innovation to continue research and development. Scotchmer (2004) emphasizes that patent protection will prevent the second round of innovation from occurring. Greenhalgh and Rogers (2010) and Bessen and Meurer (2008) use the waste of resources caused by the maintenance price of patent certification as the reason that the patent system is not conducive to innovation.

At present, most studies on the impact of patent protection on wealth distribution refer to Garcia-Peñalosa and Turnovsky (2006). Considering that the amount of assets held by consumers is different at the beginning, this is an exogenously given wealth difference setting. Most literatures believe that strengthening patent protection has widened the gap between rich and poor, such as: Chu (2010), Chu and Cozzi (2018) and Chu and Peretto (2019). In the absence of patented R&D, Blanchard (1985) sets up an overlapping generation model of eternal youth with a fixed mortality rate. Because of the wealth differences across generations, different policies trigger the redistribution of wealth across generations. This paper adopts an overlapping intergenerational model setting, wealth differences come from different birth periods, and consumer wealth differences are endogenous to the model. In addition, existing literature discusses the impact of patent protection on wealth distribution, but none of them go further to seek the optimal level of patent protection and compare it with the current level of protection in various countries. Without considering the issue of patent protection, Basu and Getachew (2020) point out that R&D policies can promote growth by creating incentives, but may exacerbate economic inequality. Through the trade-off relationship between efficiency and equity, the optimal public R&D expenditure can be obtained, and the theoretical optimal policy can be compared with the actual data in the United States.

The setting of a fixed mortality rate is not in line with the actual situation, so Heijdra and Romp (2008, 2009) developed a model in which the mortality rate increases with age. In the overlapping generation model, considering the problem of increasing mortality, most of them are used to explore the impact on the labor market. Heijdra and Mierau (2009) analyze the impact of population issues on workers' early retirement decisions. Guerra et al. (2018a) explore how population aging affects economic growth rates by affecting the proportion of dependents. Guerra et al. (2018b) estimate how mortality changes consumers' choices between work, school, and employment. Pereira (2019) studies the impact of population aging on retirement age and impact on economic growth. In addition, in the overlapping generation model, the focus of the analysis is to emphasize how various overall policies lead to different policy effects through intergenerational wealth redistribution, for example: Toda (2014), Toda and Walsh (2015), Benhabib et al. (2016), and Gabaix et al. (2016). In addition, Gârleanu et al. (2012), Gârleanu and Panageas (2015) examine the issue of asset pricing, Prettner and Canning (2014) examine the issue of retirement, and Petrucci (2002) examines the impact of consumption taxes on economic growth. The setting of the incompleteness of the annuity market mainly discusses how the imperfect competition in the annuity market changes the effect of macroeconomic policies. For example, Miyoshi and Toda (2017) argue that imperfect competition in the annuity market affect the growth effect of government transfer payments. Davidoff et al. (2005) find that there is demand for annuity insurance under imperfect competition annuity insurance, but not all of them. Some literature studies how annuity insurance affects people's consumption decisions. Bütler (2001) analyzes how annuity insurance affects people's consumption by affecting labor supply in a representative individual model. Hansen and İmrohoroğlu (2008) compare public consumption decisions in a perfectly competitive annuity insurance market with a market without annuity insurance. Chai et al. (2011) analyze the impact of annuity insurance on people's retirement decisions through a partial equilibrium model. Heijdra and Mierau (2012) analyze the effect of imperfect competition annuity insurance on economic growth in an overlapping generation model of mortality increase.

This paper uses reasonable parameter estimates to show that the optimal patent term is much higher than the current actual patent term without considering the probability of death. In the overlapping generation model with a fixed mortality rate, although the optimal patent term is shortened, it is still higher than the current actual patent term. According to the calculation of the model of increasing mortality, the optimal patent protection period is about 22 to 24 years, which is very close to the current protection period of various countries. According to the results analyzed in this paper, under the overlapping generation model, not all welfare losses of overprotection are lower than those of under-protection, and in the case of rising mortality, the extension of patents will greatly expand the intergenerational gap between rich and poor. Extending the term of patents can bring dynamic benefits by stimulating research and development. In the case of increasing mortality rate, the older generation has a high mortality rate, which can improve the welfare of the older generation. However, the younger generation owns less R&D assets, and extending the patent term does not improve the welfare of young people, resulting in a lower optimal level of protection under increasing mortality. Although patent term extension increases the wealth of the elderly, as mortality increases, the number of elderly survivors decreases, resulting in a smaller number of surviving elderly people able to obtain the patent wealth increased by patent term extension. This means that expanding patent will greatly increase the wealth of a few people, thereby further widening the gap between rich and poor. Since the increase in mortality rate does exist in the real society, in the face of the increasingly serious problem of uneven wealth distribution, analyzing the impact of patent length on intergenerational wealth redistribution has certain significance for the government to plan the length of patents.

This section is the preface, the second section sets up the basic model, the third section simulates the optimal patent length and the corresponding wealth distribution, and the fourth section is the conclusion.

2. Basic Model

In order to find the optimal patent length, the demand side in this paper follows the overlapping generations model of increasing mortality proposed by Heijdra and Romp (2008, 2009), and the supply side in this paper follows the R&D endogenous growth framework proposed by Rivera-Batiz and

Romer (1991) and takes the patent protection period into consideration. Kwan and Lai (2003) calculate the optimal patent length under a representative individual model. This paper finds the optimal patent length when mortality increases with age and compares it with Kwan and Lai (2003). Due to the complexity of this model, a table of variables and parameters is organized in Appendix 1.

The demand side follows the age-growth mortality model set by Heijdra and Romp (2008, 2009), assuming that the utility of an individual is related to the quantity of commodity consumption, and assuming that the instantaneous death probability $m(s)$ faced by an individual is related to age s per unit time, $m'(s) > 0$, maximize the present value of life cycle expected utility. Therefore, the cumulative death probability of an individual from birth to age s is $M(s) = \int_0^s m(\tau)d\tau$. In addition, we assume that the number of people born at a specific time v is β, assuming $0 \leq \beta < 1$. When $m(s) = \beta$, the model degenerates to Blanchard (1985) with a fixed death rate. Then the utility maximization choice behavior at time t for a person born at time v can be expressed as follows:

$$\max u(v,t) = \int_t^{\infty} \ln c(v,\tau) e^{-[p(\tau-t)+M(\tau-v)-M(t-v)]} d\tau, \tag{1a}$$

$$\text{s.t. } \dot{a}(v,\tau) = [r(\tau) + m(\tau - v)]a(v,\tau) + w(\tau) - c(v,\tau). \tag{1b}$$

In the above equations, ρ is the time preference rate, $u(v, t)$ is the welfare level at time t for a person born at time v (that is, the discounted value at time t of future utility of people born at time v), $c(v, \tau)$ is the consumption value at time τ for a person born at time v, $a(v, \tau)$ is the asset wealth at time τ held by a person born at time v, $r(\tau)$ is the interest rate level at time τ, $w(\tau)$ is the salary level at τ. A superscript "·" above a variable indicates the time change of that variable.

According to the setting of Heijdra and Romp (2008, 2009), the probability that a person born at time v survives at time t and is still alive at future time τ is $e^{-[M(\tau - v) - M(t - v)]}$, so equation (1a) represents the maximization of the discounted value of the person's lifetime expected utility. In addition, we follow the setting of Yaari (1965), assuming that a person can purchase an actuarial note from an insurance company, which guarantees that he can receive a fixed survival premium for each unit of assets he holds during his future survival, and once if the person dies at a specific moment, all his assets are owned by the insurance company. Since Yaari (1965) used the annuity market to deal with the deceased's estate, this series of overlapping generation models (indeterminate life) have included the annuity market, allowing consumers to deal with inheritance issues that may arise in uncertain times through the annuity market. Without an annuity market, we need to consider other approaches to deal with the estates of consumers with uncertain life spans. Assuming that the insurance market is a perfect competition, the remuneration premium is equal to the person's instantaneous death probability $m(\tau - v)$. Therefore, the equation (1b) is the person's budget constraints.

According to the above-mentioned decisions of consumers, Appendix 2 proves that the solutions for consumption growth rate and consumption function of consumers can be derived as follows:

$$\frac{\dot{c}(v,\tau)}{c(v,\tau)} = r(\tau) - \rho, \tag{2a}$$

$$c(v,\tau) = [m(\tau - v) + \rho][a(v,\tau) + h(v,\tau)], \tag{2b}$$

where,

$$h(v,\tau) = \int_{s=\tau}^{\infty} w(s) e^{-[r(s-\tau)+M(s-v)-M(\tau-v)]} \, ds. \tag{2c}$$

$h(v, \tau)$ is the human wealth at time τ for a person born at time v, which is the discounted sum of expected future wages. Therefore, $[a(v, \tau) + h(v, \tau)]$ can be regarded as the total wealth at time τ for a person born at time v.

Since the number of people born at a particular time v is β, and the probability of that a person born at time v survives at time t is $e^{-M(t-v)}$, the number of people born at time v and alive at time t is $\beta e^{-M(t-v)}$, so the aggregate consumption $C(t)$ at time t, the aggregate human wealth $H(t)$ at time t, and the aggregate asset wealth $A(t)$ at time t are respectively:

$$C(t)=\int_{-\infty}^{t}\beta\, e^{-M(t-v)}c(v,t)\,dv, \tag{3a}$$

$$H(t)=\int_{-\infty}^{t}\beta\, e^{-M(t-v)}h(v,t)\,dv, \tag{3b}$$

$$A(t)=\int_{-\infty}^{t}\beta e^{-M(t-v)}a(v,t)\,dv. \tag{3c}$$

In order to keep the total population fixed, we assume: $\int_{-\infty}^{t}\beta e^{-M(t-v)}dv=1$.

The diversified R&D model of intermediate goods proposed by Romer (1990) can be divided into two types according to the different ways of creating new varieties of intermediate products: Romer (1990) is a knowledge-driven R&D model that uses R&D technology and labor to create new intermediate goods, and Rivera-Batiz and Romer (1991) use the final goods to create new varieties of intermediate goods in the laboratory R&D model. This paper follows Kwan and Lai (2003) in using the Rivera-Batiz and Romer (1991) model as a basis for discussing optimal patent terms. Therefore, on the supply side, this paper still uses the model setting of Rivera-Batiz and Romer (1991) to consider the form of technological progress of diversified intermediate products. There are three kinds of firms in the economic system: one is the final goods sector of perfect competition; the other is the R&D sector of monopolistic competition; the third is the intermediate goods sector of perfect competition. The patent issue is not considered in Romer (1990) and Rivera-Batiz and Romer (1991). Once the innovation is successful, R&D firms can exclusively supply intermediate goods permanently. Therefore, the R&D market is a monopolistic competition. Both this paper and Kwan and Lai (2003) consider the patent issue. After the patent expires, R&D firms cannot exclusively provide intermediate goods, so a perfectly competitive intermediate goods market emerges after the patent expires.

Since the final goods market is perfectly competitive and the firm is the price taker, in order to simplify the analysis, the final goods price is united as one. In addition, in order to simplify the notation, the following inferences in this paper are only marked with the subscript "t" representing time where necessary, otherwise it will be omitted. Therefore, the profit maximization choice behavior of the final goods firm can be expressed as:

$$\max_{\{Y,L,x(j)\}} \quad Y - wL - \int_0^N P(j)\,x(j)dj, \tag{4a}$$

$$\text{s.t. } Y = BL^{1-\alpha} \int_0^N x(j)^\alpha dj. \tag{4b}$$

In the above equations, Y is the output of final goods, w is the wage rate, L is the labor input, $P(j)$ is the price of the jth intermediate goods, $x(j)$ is the input amount of the jth intermediate goods, and B is the technology parameter. There are N kinds of continuous intermediate goods. Equation (4a) indicates that the final goods firm pursues the maximum profit, and the constraint condition of the equation (4b) is the production function of the final goods.

According to the above-mentioned decision-making of the final goods firm, the solutions for the final goods firm's employment of labor and the use of various intermediate goods can be obtained as follows:

$$(1-\alpha)L^{-\alpha} \int_0^N x(j)^\alpha\, dj = w, \tag{5a}$$

$$\alpha L^{1-\alpha} x(j)^{\alpha-1} = P(j). \tag{5b}$$

The R&D department belongs to the market structure of monopolistic competition, and innovative firm only need to invest η units of final goods to develop one new intermediate goods. Once a successful innovative firm develops a new intermediate goods, the government guarantees that the innovator can monopolize the supply of this intermediate goods within T period, and every investment in a final goods in each period can produce a unit of intermediate goods. According to the above setting, the profit maximization choice behavior of innovative firm can be expressed as:

$$\max_{\{x^m(j),P^m(j)\}} \quad \mathrm{E}[V(t,t)]-\eta, \tag{6a}$$

$$\text{s.t. } \mathrm{E}[V(t,t)]=\int_t^{t+T}\pi^m(j)e^{-\int_t^{\tau}r(z)dz}d\tau, \tag{6b}$$

$$\pi^m(j)=[P^m(j)-1]x^m(j), \tag{6c}$$

$$x^m(j)=\left[\frac{\alpha\beta}{P^m(j)}\right]^{\frac{1}{1-\alpha}}L. \tag{6d}$$

In the above equations, the expected value at time s of a patent developed at time t is $\int_{-\infty}^{t}\beta e^{-M(t-v)}dv=1$

$\mathrm{E}[V(t,s)]=\int_s^{t+T}\pi^m(j)e^{-\int_s^{\tau}r(z)dz}d\tau$, $\pi^m(j)$ is the single-period profit of innovative firm selling intermediate goods, and $x^m(j)$ is the quantity of intermediate goods produced by innovative firms in each period, $P^m(j)$ is the pricing of intermediate goods by innovative firms. Equation (6a) expresses that the innovative firm pursues the maximum profit, the constraint condition of equation (6b) sets the expected value of R&D patents, the constraint condition of equation (6c) sets the single-period profit of the innovative firm selling intermediary goods, and the constraint condition of equation (6d) is the demand function of final goods firm for this kind of intermediate goods. Among them, the discounted value of the total expected profits that successful innovative firms can earn during the period of monopoly supply of intermediate goods is the expected value of R&D patents.

According to the above decision of the innovative firm, the solutions for pricing and output of the innovative firm's sale of intermediate goods can be obtained:

$$P^m(j)=\frac{1}{\alpha}=P^m>1, \tag{7a}$$

$$x^m(j)=(\alpha^2B)^{\frac{1}{1-\alpha}}L=x^m. \tag{7b}$$

According to equations (6b), (6c), (7a), and (7b), Appendix 3 demonstrates the value of R&D patents can be obtained as:

$$\mathrm{E}[V(t,t)] = \frac{(1-\alpha)(\alpha^2 B)^{\frac{1}{1-\alpha}} L[1-e^{-rT}]}{r\alpha}. \tag{8a}$$

On the other hand, because the innovation of the R&D department belongs to the market structure of monopolistic competition, innovative firms do not have excess profits. Therefore, E[$V(t, t)$] = η must be established. According to equation (8a), the equilibrium market interest rate is:

$$r(t) = \frac{L(\alpha^2 B)^{\frac{1}{1-\alpha}}(1-\alpha)[1-e^{-rT}]}{\alpha\eta} = r. \tag{8b}$$

Equation (8a) shows that increasing the length of patents increases the value of patents, and equation (8b) shows that increasing the length of patents increases market interest rates. Since the R&D department belongs to the market setting of competition, in the long run, the R&D department does not make profits, so the value of R&D will be equal to the cost of R&D. If equation (8b) is substituted into equation (8a), the R&D cost is not affected by the length of the patent, which cannot be understood as the length of the patent cannot affect the value of R&D. Combining equations (8a) and (8b), increasing the length of the patent increases the value of the patent, which in turn makes the R&D department profitable. Under the competitive market setting, it drives new R&D departments to enter the market, leading to an increase in the demand for funds, and finally after the interest rate rises, the value of R&D decreases, returning to the state where the R&D department earns normal profits, and a new equilibrium is reached. When E[$V(t, t)$] = η, if the supply of funds (savings) is greater than the demand for funds (R&D), the excess supply in the funds leads to a decline in interest rates, which further increases the value of R&D (E[$V(t, t)$]), so that E[$V(t, t)$] > η, R&D is profitable, and new R&D departments enter the market, increasing the demand for funds until the supply and demand of the funds market reach a balance. In addition, when E[$V(t, t)$] = η, if the supply of funds (savings) is less than the demand for funds (R&D), the excess demand in the funds market triggers an increase in interest rates, further reducing value of R&D (E[$V(t, t)$]), so that E[$V(t, t)$] < η, and then there is a loss in R&D, which reduces the R&D departments in the market and reduces the demand for funds until the supply and demand of the funds market reaches a balance. Therefore, in equilibrium, E[$V(t, t)$] = η and the equilibrium of the funds market must be established at the same time, and the equilibrium interest rate is determined by the equilibrium of the funds market, but this interest rate must make E[$V(t, t)$] = η be established. Otherwise, if E[$V(t, t)$] ≠ η, the funds market has to change due to the increase or decrease of R&D departments, which means that the funds market has not yet reached equilibrium. A detailed description of this section can be found on pages 292 to 295 of Barro and Sala-i-Martin (2004).

Since a successful innovation firm develops a new intermediate good, it can monopolize the supply of this intermediate good for T period. After the T period, this intermediate good will be supplied by perfectly competitive firms. Therefore, the category $N^c(t)$ of intermediate goods supplied by perfectly competitive firms in period t is the total category $N(t-T)$ of all intermediate goods in period $t-T$:

$$N^c(t) = N(t-T). \tag{9a}$$

Since every unit of intermediate goods produced must be invested in a unit of final goods as the input of production, and in a perfectly competitive market, the firm's profit is always 0. Therefore, the price of intermediate goods supplied by perfectly competitive firms is equal to the price of final goods. If P^c is the price of intermediate goods supplied by perfectly competitive firms, then $P^c = 1$, and the quantity x^c of intermediate goods supplied by perfectly competitive firms is:

$$x^c = (\alpha B)^{\frac{1}{1-\alpha}} L. \tag{9b}$$

The output of the final goods is:

$$Y = BL^{1-\alpha}[N^c(x^c)^\alpha + (N - N^c)(x^m)^\alpha]. \tag{9c}$$

Market equilibrium is determined by supply equaling demand. In the labor market, the supply of labor is provided by consumers, and the demand for labor is determined by the final goods firms. In the funds market, the savings of consumers is the supply of funds, and the R&D expenses required by R&D departments are the demand for funds. In a competitive market, once the R&D value is realized, the R&D department immediately repays the loan, resulting in the need for funds only to preserve the unrecovered R&D value. Also, in the final goods market, the supply of the final goods is provided by the final goods firms. As for the demand for final goods, part of the demand for final goods is consumer consumption, part is the production factors of R&D departments, and part is the production input of intermediate goods. Thus, the equilibrium conditions for labor, funds, and final goods markets are:

$$L = 1, \tag{10a}$$

$$A(t) = \int_{t-T}^{t} \dot{N}(s) E[V(s,t)] ds, \tag{10b}$$

$$Y = C + \eta \dot{N} + N^c x^c + (N - N^c) x^m. \tag{10c}$$

The zero-profit condition for R&D firm is that the firm can repay all principal and interest during the profit-making period. The fund provider can earn each period's interest through the fund, while living benefits come from compensation from insurance companies. In an annuity insurance provided by a perfectly competitive insurance company, people receive living premiums during their lifetime, and the funds held after death belong to the insurance company. If survival benefits equal mortality, the insurance company can earn a normal profit. In the fund market, the fund demand of all R&D firms is equal to the fund supply of all people, while in the insurance market, everyone's insurance needs are provided by insurance companies. However, since time is continuous, some R&D firms have repaid all the funds, and there are also new R&D firms that need funds to successfully develop new products. The public does not need to recover all the lent funds, because some firms have repaid all the funds and the market is completely competitive, there are too many participants, and the supply and demand relationship of funds in the macroeconomic market continues to exist. As long as the supply and demand in each period can maintain settlement, the market can reach equilibrium.

Macroeconomic equilibrium is determined by the constraints and choices of all consumers and firms and the equilibrium conditions of all markets. When the macroeconomy reaches a steady state of equilibrium growth, the growth rate of all macroeconomic variables such as aggregate consumption (C), aggregate human wealth (H), aggregate asset wealth (A), the wage rate (w), perfectly competitive types of intermediate goods (N^c), all types of intermediate goods (N), and final output (Y) will converge and the ratio of any two macro variables will converge to a fixed value. The growth rate when all macroeconomic variables reach a steady state is called the balanced growth rate (g). In the long run, 12 equations such as (3a), (3b), (3c), (5a), (7a), (7b), (8b), (9a), (9b), (9c), (10a), and (10c) will jointly determine 12 macro variables such as C/N, H/N, A/N, w/N, N^c/N, Y/N, P^m, x^m, x^c, L, g, and r. However, the growth rate of personal consumption ($c(v, t)$) and assets ($a(v, t)$) will not equal the equilibrium economic growth rate. Under a person's solutions, given the assets held at the beginning of the period, the growth rate of personal consumption and assets in each period are determined by equations (1b) and (2a).

3. Optimal Patent Length

This section seeks the optimal patent length that maximizes social welfare. First, we define the welfare of each generation, analyze the impact of extending the patent term on each generation, and then summarize the welfare of each generation into macro welfare levels. Further analyze the optimal patent length to maximize macro-social welfare. Finally, numerical analysis is used to find the optimal patent length under various circumstances, the intergenerational wealth gap under the optimal patent length, and the loss of social welfare when the patent length is too long and too short.

According to equation (1a), considering that the death rate increases with age, when the economic system reaches a long-term equilibrium,

Appendix 4 demonstrates the welfare level at current time t for a person born at time v in the past is:

$$u(v,t) = \int_t^\infty \ln c(v,t) e^{-[\rho(\tau-t)+M(\tau-v)-M(t-v)]} d\tau$$
$$\cong \frac{\ln c(v,t)}{m(t-v)+\rho} + \frac{r(t)-\rho}{[m(t-v)+\rho]^2}. \tag{11a}$$

Where,

$$c(v,t) = [m(t-v)+\rho][a(v,t)+h(v,t)], \tag{11b}$$
$$h(v,t) = \int_{\tau=t}^\infty w(\tau) e^{-[r(\tau-t)+M(\tau-v)-M(t-v)]} d\tau. \tag{11c}$$

Appendix 4 demonstrates the welfare level at current time t for a person born at time s in the future is:

$$u(s,t) = e^{-\delta(s-t)} u(s,s) \cong e^{-\delta(s-t)} \left[\frac{\ln c(s,s)}{\beta+\rho} + \frac{r(s)-\rho}{(\beta+\rho)^2} \right], \tag{12a}$$

where,

$$c(s,s) = c(t,t) e^{g(s-t)} = (\beta+\rho) H(s). \tag{12b}$$

δ is the discount rate of welfare for future generations. Since the current survivors use the time preference rate to discount their future consumption, and the generations born in the future only start consuming when they are born, therefore, if we want to consider the current value of the welfare of future generations, we should also discount it, and their time preference rate must be consistent with the existing generations, so that the welfare of each generation has the same weight and does not cause the time-inconsistency problem referred to by Calvo and Obstfeld (1988). Therefore, Calvo and Obstfeld (1988) set $\delta = \rho$, so this paper also continues this setting in the subsequent numerical analysis.

From equations (11a) and (12a), the impact of extending the patent protection period (T) on the welfare at the current time t for a person born at time v in the past and at time s in the future are respectively:

$$\frac{du(v,t)}{dT} \cong \frac{1}{[m(t-v)+\rho]^2}\left[\underbrace{\frac{\partial r(t)}{\partial T}}_{\text{Real interest rate effect}}\right] + \frac{1}{[m(t-v)+\rho]}\left[\underbrace{\frac{1}{c(v,t)}\frac{\partial c(v,t)}{\partial T}}_{\text{Private consumption effect}}\right] \gtreqless 0, \quad (13a)$$

$$\frac{du(s,t)}{dT} \cong \frac{e^{-\delta(s-t)}}{(\beta+\rho)^2}\left[\underbrace{\frac{\partial r(s)}{\partial T}}_{\text{Real interest rate effect}}\right] + \frac{e^{-\delta(s-t)}}{(\beta+\rho)}\left[\underbrace{\frac{1}{c(s,s)}\frac{\partial c(s,s)}{\partial T}}_{\text{Private consumption effect}}\right] \gtreqless 0, \quad (13b)$$

where,

$$\frac{\partial c(s,s)}{\partial T} = e^{g(s-t)}\frac{\partial c(t,t)}{\partial T} + c(s,s)(s-t)\frac{\partial g}{\partial T}. \quad (13c)$$

Equations (13a) and (13b) show that the marginal effect of extending patent length on the welfare of those born in the past (v) and in the future (s) is uncertain. Both affect the welfare of each generation through the following two effects: (1) Real interest rate effect: extending the patent length increases the welfare of each generation by increasing the real interest rate. In equations (13a) and formula (13b), the first term on the right represents the real interest rate effect. (2) Private consumption effect: extending the length of patents increases the value of R&D patents, thereby increasing the value of savings ($a(v, t)$); in addition, extending the length of patents increases the types of intermediate goods that can be monopolized, reducing the average amount of intermediate goods supply also reducing the marginal product of labor and wages, thereby reducing human wealth ($h(v, t)$). According to equation (11b), people's consumption is a proportion of their total wealth, and the total wealth of individuals includes asset wealth ($a(v, t)$) and human wealth ($h(v, t)$). Older people with higher mortality accumulate more savings and hold less human wealth, while extending the patent term increases the total wealth of the elderly, thereby increasing their consumption; young people with lower mortality have relatively less savings and more human wealth, while extending patent terms reduces the total wealth of young people, thereby discouraging their consumption. Furthermore, future generations hold no savings assets, therefore extending patent terms reduces human wealth ($h(t, t)$) for the current generation due to lower output of intermediate goods. The human wealth of future generations $h(s, s) = h(t, t) e^{g(s-t)}$ also decrease due to the decline in the output of intermediate goods, but patent extension can increase the human wealth of future generations by increasing economic growth. According to formula (12b), the consumption of the future generation is a proportion of its human wealth. For the more recent future generations, the effect of economic growth on human wealth is small, and extending the length of patents definitely reduces their human wealth, leading to a decline in consumption for this generation. However, for the more distant future generations, economic growth has a greater impact on human wealth, and extending the patent term may increase their human wealth, which leads to an increase in consumption by this generation. Therefore, the impact of extending the patent term on consumption and welfare in each generation cannot be determined. In equations (13a) and (13b), the second term on the right side of the equations are the private consumption effect. Because extending the patent length increases the consumption of the older and more distant future generations, both the real interest rate effect and the private consumption effect increases the welfare of the older and more distant future generations; however, extending the patent length depresses the consumption of younger and more recent future generations. Although the real interest rate effect can increase the welfare of younger and more recent future generations, the private consumption effect depresses the welfare of younger and more recent future generations, and finally may lead to extended patent lengths that are not conducive to younger and more recent future generations. Therefore, the impact of extending patent length on the welfare of each generation cannot be determined in equations (13a) and (13b).

After understanding the impact of patent length on the welfare of individual generations, we need to analyze the impact of patent length on macroeconomy social welfare, which is the sum of the welfare of all generations after considering the population weight. To allow the surviving population from the past generations to the present, following Bovenberg and van Ewijk (1997) and Pautrel (2008), the welfare of the current generations at time *t* can be set as:

$$U(t)=\int_{-\infty}^{t}\beta e^{-M(t-v)}u(v,t)dv. \tag{14a}$$

Following Calvo and Obstfeld (1988) and Mathieu-Bolh (2006), the present value of future generations' welfare considers time preference discounting. Referring to Bovenberg and van Ewijk (1997), the number of births per period (β) is the basis for the welfare of future generation. The welfare of future generations at time *t* can be shown that:

$$U^{F}(t)=\int_{t}^{\infty}\beta e^{-\delta(s-t)}u(s,s)ds. \tag{14b}$$

According to equations (14a) and (14b), the social welfare function $SW(t)$ at time *t* is:

$$SW(t)=U(t)+U^{F}(t). \tag{14c}$$

Differentiating equations (14a) and (14b) with respect to *T*, we have the following effects of extending the patent term on the welfare of current and future generations in the overlapping generation model:

$$\frac{dU(t)}{dT}=\int_{-\infty}^{t}\beta e^{-M(t-v)}\frac{du(v,t)}{dT}dv\gtreqless 0, \tag{15a}$$

$$\frac{dU^{F}(t)}{dT}=\int_{t}^{\infty}\beta e^{-\delta(s-t)}\frac{du(s,s)}{dT}ds\gtreqless 0. \tag{15b}$$

Equations (15a) and (15b) show that the marginal impact of *T* on the total welfare of the current and future generations is uncertain. According to equation (14c), we have:

$$\frac{dSW(t)}{dT}=\int_{-\infty}^{t}\beta e^{-M(t-v)}\frac{du(v,t)}{dT}dv+\int_{t}^{\infty}\beta e^{-\delta(s-t)}\frac{du(s,s)}{dT}ds\gtreqless 0. \tag{15c}$$

Equation (15c) shows that the marginal effect of T on the overall social welfare (including the welfare of the existing generation and the future generation) is uncertain. From equation (13a) and equation (13b), we can see that extending the patent length has different effects on the welfare of different generations. Extending the patent length increases the welfare of the older generation and the more distant future generation, and reduce the welfare of the younger generation and the more recent future generation. Therefore, extending the patent length increases the welfare of the older generation and the more distant future generations, resulting in marginal benefits to the overall social welfare. It also reduces the welfare of the younger generations and the more recent future generations, resulting in a marginal loss to the overall social welfare. The older generation has accumulated more patent assets, more wealth and higher consumption. The longer future generations have accumulated a longer period of economic growth, and they can have higher wealth and higher consumption at birth. When the patent length is relatively short, the gap between high-consumption generations (including older generations and more distant future generations) and low-consumption generations (including younger generations and more recent future generations) is small. The increased utility of high-consumption consumers due to increased consumption may exceed the reduced utility of low-consumption consumers due to reduced consumption. Therefore, extending the patent length can improve the welfare of the whole society. Conversely, when the length of the patent is relatively long and the gap between the high-consumption generation and the low-consumption generation is large, under the setting of diminishing marginal utility, the increase in the utility of high consumers due to increased consumption by extending the patent length is smaller than that of low consumers due to decreased consumption. Therefore, reducing the length of patents can reduce the welfare of the whole society. Likewise, due to diminishing marginal utility, extending the patent term increases the welfare of the high-consumer generation (by increasing the consumption of the high-consumption generation) diminishingly, while decreasing the welfare of the low-consumption generation (by reducing the consumption of the low-consumption generation) is incremental. Finally, there exists an optimal patent length such that the marginal benefit of increasing the welfare of the high-consumption generation is equal to the marginal loss of reducing the welfare of the low-consumption generation. When $dSW(t)/dT = 0$ in equation (15c) holds, the marginal loss is equal to the marginal gain.

To illustrate the result of the model regarding the optimal T, this section conducts a numerical analysis. Table 1 lists the parameter values selected in this paper. Gompertz (1825) proposes a model in which the mortality rate increases exponentially with age, and Makeham (1860) adds an age-independent constant to the mortality rate. We set the mortality function under the increasing mortality rate as follows:

$$m(t-v) = \mu_0 + \mu_1 e^{\mu_2 (t-v)}, \tag{16a}$$

$$M(t-v) = \mu_0 (t-v) + \frac{\mu_1}{\mu_2}[e^{\mu_2 (t-v)} - 1]. \tag{16b}$$

Equation (16a) is often called the Gompertz-Makeham law, where, μ_0 is the age-independent exogenous mortality rate, μ_1 is the age-affected mortality rate of newborns, and μ_2 is the aging rate (rate of aging), that is, the rate of increase in mortality rate with age, in equation (16a), $\mu_1 e^{\mu_2 (t-v)}$ belongs to age-related mortality. This paper refers to Table 1 on page 101 of Heijdra and Romp (2008), which uses the 2006 Dutch Population Mortality Statistical Database to estimate the parameter values of the mortality rate increasing with age: $\mu_0 = 0.002437$, $\mu_1 = 0.0000552$, and $\mu_2 = 0.0964$, and assume $m(t-v) = \mu_0 + \mu_1 e^{\mu_2 (t-v)}$. Heijdra and Romp (2008) used data from the 2006 Dutch Population Survey to estimate birth rates below the global average, so this paper considers twice the birth rate for comparison. The result of twice the birth rate after rounding is 0.005, so 0.005 is considered the high birth rate. In addition, this paper considers the total population to be fixed ($\beta = \mu_0 + \mu_1$) and sets the death rate to be fixed ($\mu_0 = \mu_1 = 0$) to facilitate comparison with the increasing death rate. Finally, if future generations are not considered, the welfare discount rate for future generations approach infinity ($\delta \rightarrow \infty$); if future generations are considered, $\delta = \rho$ satisfies the temporal consistency condition of Calvo and Obstfeld (1988). For each case, we set the parameters β, μ_0, μ_1, μ_2 and δ as in Table 2.

Table 1 Benchmark

Definition	Parameter	Value	Reference
Capital share	α	0.625	Kwan and Lai (2003), Hall (1986)
Time preference rate	ρ	0.05	Lucas (1988), Prescott (1986), Kwan and Lai (2003), Zeng and Zhang (2007)
Technology parameter	B	1	Lucas (1988), Prescott (1986), Kwan and Lai (2003), Zeng and Zhang (2007)
R&D cost	η	0.65	Kwan and Lai (2003)
Total amount of intermediate goods at the beginning of the period	$N(t)$	1	Kwan and Lai (2003), Zeng and Zhang (2007)

Source: Compiled by this study.

Table 2 Case Parameters

Case Model	Birth Rate	Death Rate	Future Generations	β	μ_0	μ_1	μ_2	δ
(1)	Representative individual			0	0	0	0	$\to\infty$
(2)	Low	Fixed	without	0.002437	0.002437	0	0	$\to\infty$
(3)	Low	Fixed	including	0.002437	0.002437	0	0	0.05
(4)	Low	Increasing	without	$\mu_0+\mu_1$	0.002437	0.0000552	0.0964	$\to\infty$
(5)	Low	Increasing	including	$\mu_0+\mu_1$	0.002437	0.0000552	0.0964	0.05
(6)	High	Fixed	without	0.005	0.005	0	0	$\to\infty$
(7)	High	Fixed	including	0.005	0.005	0	0	0.05
(8)	High	Increasing	without	0.005	$\beta-\mu_1$	0.0000552	0.0964	$\to\infty$
(9)	High	Increasing	including	0.005	$\beta-\mu_1$	0.0000552	0.0964	0.05

Source: Compiled by this study.

From Table 3, we can obtain the following important conclusions:

(1) The optimal patent length obtained by Kwan and Lai (2003) using a representative individual model ignores generational wealth differences. After comparing representative individual model with the fixed mortality rate, this paper finds that shortening the patent term reduces the consumption and welfare of the elderly and increases the consumption and welfare of the young. Due to diminishing marginal utility, the welfare of the elderly decreases lower than the increased welfare of young people, thus increasing the total welfare of young people. Therefore, the optimal patent length that maximizes the total welfare of the current generation under fixed mortality is lower than that of the representative individual model.

(2) In the representative individual model used by Kwan and Lai (2003), different generations cannot be distinguished, and therefore the optimal patent length with or without considering offspring cannot be compared as in this paper. Extending the length of patents is equivalent to transferring part of the wealth of the young to the elderly. The wealth of the older generation is greater than that of the younger generation under a fixed mortality rate. Extending the length of patents expands the inter-generational wealth gap. This means that the extension of the patent term makes the wealth of the young born relative to the old decline. Although unborn offspring will also reduce human wealth due to the extension of patent length, the extension of patent length also improves economic growth, and the human wealth of future generations at birth will increase due to high economic growth. Therefore, the human wealth of future generations may not be reduced by extension of patent term. Since the high real interest rate and high economic growth brought about by the extension of the patent term increases people's future consumption, future generations benefit more from the extension of the patent term. As a result, compared with the overlapping generation model that does not consider the future generations, the optimal patent length corresponding to the overlapping generation model that considers the future generations is longer.

(3) In the representative individual model used by Kwan and Lai (2003), the issue of mortality is not considered, so it is impossible to compare the optimal patent length with or without considering the increase in mortality with age as in this paper. Extending the length of patents increases the value of successful R&D. Since the older generation holds more R&D assets, extending the length of patents makes the older generations richer and increases the economic growth rate. In the case of increasing death rates, the mortality rate of the older generation is very high, and future economic growth will not be able to improve the welfare of the elderly. Because the younger generation holds less R&D assets, extending the length of patents cannot improve the welfare of the young population, resulting in a lower optimum level of protection in the increasing mortality rate model.

Table 3 Optimal Patent Length

Model	Intergenerational Wealth Difference	Optimal Patent Length	Welfare Loss when Patent Length Exceeds Optimal One Year	Welfare Loss when Patent Length is Less than Optimal One Year
(1) Representative individual	0 (0)	41 years	0.0007	0.0022
(2) Low birth rate fixed death rate without future generations	0.13848 (0.01241)	30 years	0.0093	0.0022
(3) Low birth rate fixed death rate including future generations	0.13848 (0.01241)	30 years	0.0031	0.0101
(4) Low birth rate increasing mortality rate without future generations	0.00206 (0.02144)	22 years	0.0276	2.6126
(5) Low birth rate increasing mortality rate including future generations	0.03511 (0.01964)	24 years	0.0045	0.0002

Table 3　Optimal Patent Length (continued)

Model	Intergenerational Wealth Difference	Optimal Patent Length	Welfare Loss when Patent Length Exceeds Optimal One Year	Welfare Loss when Patent Length is Less than Optimal One Year
(6) High birth rate fixed death rate without future generations	0.11840 (0.01463)	28 years	0.0020	0.0109
(7) High birth rate fixed death rate including future generations	0.13038 (0.01379)	29 years	0.0021	0.0130
(8) High birth rate increasing mortality rate without future generations	0.00212 (0.02200)	22 years	0.0487	4.1482
(9) High birth rate increasing mortality rate including future generations	0.02718 (0.02055)	23 years	0.0001	0.0089

Note: 1. The intergenerational wealth difference in the table is the proportion of the new generation's total wealth below the average.
2. The numbers in parentheses represent the increase in intergenerational wealth differences caused by extending patent terms by one year.

(4) The representative individual model used by Kwan and Lai (2003) does not consider the birth rate issue, and therefore cannot explore the impact of birth rate on optimal patent length as in this paper. The higher the birth rate, the more obvious the intergenerational wealth difference, and the extension of the patent term reduces more social welfare due to the expansion of the wealth difference, resulting in the shorter the optimal patent term.

(5) Currently Taiwan, China, Japan, South Korea, Australia, the United States, Canada, European patent member countries, the United Kingdom, Germany, France, the Netherlands, and Italy all use 20 years as the protection period for invention patents. Even considering factors that patent detract from social welfare, such as intergenerational wealth

differences and increased mortality with age, at present, most countries still have insufficient patent lengths. Compared with the representative individual model used by Kwan and Lai (2003), the optimal patent length that considers the increase in mortality with age is closer to the actual patent length in each country.

(6) Kwan and Lai (2003) have another important conclusion: under overprotection, the welfare loss is very small; conversely, under under-protection, the welfare loss is very large. In Table 3, the numerical analysis of this paper finds that in the overlapping generation model, the conclusion that overprotection causes smaller welfare losses does not hold. There is no inter-generational wealth difference in the representative individual model. When the patent length is too long, only the loss of social welfare is caused by the reduction of private consumption, which leads to a small welfare loss due to over-protection. However, considering the overlapping generation structure, the expansion of intergenerational wealth differences caused by protection may aggravate the overall welfare loss of over-protection, which in turn may lead to greater welfare loss due to over-protection.

(7) The representative individual model used by Kwan and Lai (2003) cannot discuss wealth differences across generations as in this paper. Regardless of whether the death rate is increasing or not, extending the length of patents increases inter-generational wealth differences and form inter-generational redistribution of wealth. Compared with the situation where the death rate is fixed, the elderly who have accumulated more patent assets face a higher death rate under the increase in the death rate, so that the intergenerational wealth difference corresponding to the increase in the death rate is smaller. However, since extended patent increases the overall patent value, with a fixed mortality rate, extended patent increases the wealth of the older generation and reduce the wealth of the younger generation. Moreover, the increase in the wealth of the older generation must exceed the decrease in the wealth of the younger generation, and thus widening the wealth gap between generations. If mortality increases with age, fewer people survive as they get older. This means

that extending patent increase the wealth of a few by a large amount and decrease the wealth of the majority by a small amount. Moreover, the sum of the increases in wealth for the few must exceed the sum of the decreases in wealth for the majority. Therefore, the increase in mortality with age deepen the gap between rich and poor. Substantially increasing the wealth of the surviving elders will lead to a higher intergenerational redistribution of wealth formed by extending patents than when mortality is fixed. Since the death rate is actually rising, the intergenerational wealth difference caused by the extension of patent length is underestimated if the increase in death rate is not considered. In short, under the condition of rising mortality, the generational wealth difference is not large, but the generational wealth redistribution caused by patent extension is large.

4. Conclusion

At present, most countries use 20 years as the length of invention patents (for example: Taiwan, China, Japan, South Korea, Australia, the United States, Canada, European patent member states, the United Kingdom, Germany, France, the Netherlands, Italy, Vietnam, Thailand, Singapore and Malaysia, etc.), Kwan and Lai (2003) use Romer’s diversified intermediary technology progress model to analyze the problem of patent length, and conducted a numerical analysis on the length of patents in the United States, and find that the length of patents in the United States is lower than the optimal level. The welfare loss of patent length exceeding optimum is lower than that of patent length less than optimum, which means that the government should increase the patent length of patents. However, Kwan and Lai (2003) consider the setting of representative individuals and cannot capture the wealth difference brought about by patent extension. In this paper, referring to the overlapping generation model of death rate increasing with age set by Heijdra and Romp (2008, 2009), the estimated optimal patent length is lower. At present, the patent length of various countries, the theoretical optimal patent length is still longer than the actual length. Furthermore, welfare losses below optimal patent terms are not uniformly higher in all cases, and, holding mortality rates fixed, underestimate the widening intergenerational wealth gap caused by longer patent terms. Therefore, if the government intends to extend the patent term, it needs to be carefully evaluated. In addition to affecting economic growth as Bloom et al. (2000) and Bloom et al. (2021) pointed out, population structure also affects the growth and welfare effects of macroeconomic policies. Many macroeconomic models do not account for actual increases in mortality with age or even intergenerational differences in wealth, which can misestimate the effects of various policies.

Most current theories point out that the current actual patent term is too short and advocate extending the patent term. However, if the impact of increased mortality is considered, this paper finds that the theoretically found optimal patent length is very close to the current actual patent length. In terms of theoretical research, the existing literature does not consider the fact that mortality increases with age and overestimates the theoretical optimal patent length. This paper makes up for this theoretical flaw. From the perspective of actual policy recommendations, this paper is different from past research in advocating extending the current patent term. After considering the rising mortality rate, since the optimal patent length is close to the current level, considering the stability of the policy, this paper recommends that the patent length should be maintained at the current level. In terms of theoretical research, this paper considers the phenomenon of mortality increasing with age, which has been ignored by previous researches. From the perspective of social practice, the conclusion of this paper is that patent length should be maintained at the current level, which is different from the extension of patent length supported by past theoretical researches.

Currently, there are three kinds of patents: invention patents, utility model patents and design patents. The definition of invention patent is “the creation of technical ideas that use the laws of nature”, the utility model patent is “the use of technical concepts of the laws of nature to create the shape, structure or device of an object”, while the design patent is “the use of visual appeal to the shape, pattern of an object or the creation, combination, creation of colors or other objects”. Different patent definitions apply to different economic models. This paper follows the intermediary goods diversification R&D model set by Romer (1990), suitable for discussing invention patents but the others. Therefore, the patent discussed in this paper is only for invention patent. Utility model patents, which improve technology by creating existing goods, are suitable for discussion using the quality improvement R&D models of Grossman and Helpman (1991) and Aghion and Howitt (1992). Design patent is suitable for discussion using Dixit and Stiglitz (1977) diversified consumer goods model. The latter two patents with different economic models cannot apply to the result of this paper directly and will be left as future work.

Appendix 1

This appendix provides the definitions of all variables and parameters used in this paper.

Table A1 Definitions of all Variables and Parameters

Quantity Variables	
$c(v, \tau)$	the consumption value at time τ for a person born at time v
$C(t)$	the aggregate consumption at time t
$a(v, \tau)$	the total asset wealth at time τ held by a person born at time v
$A(t)$	the aggregate asset wealth at time t
$h(v, \tau)$	the human wealth at time τ for a person born at time v
$H(t)$	the aggregate human wealth at time t
$x(j)$	the input amount of the jth intermediate goods
$N(t)$	the number of varieties of diversified intermediate goods
$N^c(t)$	the number of horizontal intermediate goods that have become competitive at time t
$Y(t)$	the quantity of final goods
$L(t)$	the labor input
x^m	the quantity of input in the intermediate goods with various types before the patent expires
x^c	the quantity of input in the intermediate goods with various types that have become competitive
Price Variables	
$r(\tau)$	the real rate of return at time τ
$w(\tau)$	the wage rate at time τ
$P(j)$	the price of the jth intermediate goods
P^m	the price of input in the intermediate goods with various types before the patent expires
P^c	the price of input in the intermediate goods with various types that have become competitive
Profit Variables	
$\mathrm{E}[V(s, t)]$	the expected value at time s of a patent developed at time t

Table A1 Definitions of all Variables and Parameters (continued)

$\pi^{m}(j)$	the flow of monopoly profit to the owner of the jth of variety intermediate goods.
Welfare Variables	
$u(v, t)$	the welfare level at time t for a person born at time v
$U(t)$	the welfare of the current generations at time t
$U^{F}(t)$	the welfare of future generations at time t
$SW(t)$	the social welfare function at time t
Equilibrium Variable	
g	the long-term economic growth rate
Mortality Rates	
$M(s)$	the cumulative death probability of an individual from birth to age s
$m(s)$	instantaneous death probability faced by an individual is related to age s per unit time
Model Parameters	
ρ	the exogenous subjective discount rate
α	production share of intermediate goods
η	the cost to develop one new intermediate goods
δ	the discount rate of welfare for future generations
Policy Parameter	
T	the patent term

Appendix 2

This appendix derives the optimal choices of consumers.

To analyze the solutions to the dynamic optimization problem, we refer to the mathematical appendix in Barro and Sala-i-Martin (2004). We assume that the present value of the Hamiltonian is as follows:

$$J = \ln c(v,\tau) e^{-[\rho(\tau-t)+M(\tau-v)-M(t-v)]} + \lambda(\tau)\{[r(\tau)+m(\tau-v)]a(v,\tau)+w(\tau)-c(v,\tau)\}, \tag{A1}$$

where the expression in braces equals $\dot{a}(v, \tau)$ from equation (1b). The variable $\lambda(\tau)$ is the present value shadow price of asset wealth. The first-order conditions for a maximum of $u(v, t)$ are

$$\frac{\partial J}{\partial c(v,\tau)} = 0 \Rightarrow \lambda(\tau) = \frac{e^{-[\rho(\tau-t)+M(\tau-v)-M(t-v)]}}{c(v,\tau)}. \tag{A2}$$

$$\frac{\partial J}{\partial a(v,\tau)} = -\dot{\lambda}(\tau) \Rightarrow \dot{\lambda}(\tau) = -[r(\tau)+m(\tau-v)]\lambda(\tau). \tag{A3}$$

The transversality condition is

$$\lim_{\tau\to\infty} \lambda(\tau)\, a(v,\tau) = 0. \tag{A4}$$

If we differentiate equation (A2) with respect to time and substitute for $\lambda(\tau)$ from this equation and for $\dot{\lambda}(\tau)$ from equation (A3), we get the basic condition for choosing consumption over time:

$$[r(\tau)+m(\tau-v)] = [\rho+m(\tau-v)] + \frac{\dot{c}(v,\tau)}{c(v,\tau)}. \tag{A5}$$

From equation (A5), the optimal consumption growth rate equation (2a) for individual consumers can be derived.

Putting the result of equation (2a) into equation (A2) can obtain the updated transversality condition:

$$\lim_{\tau\to\infty} e^{-[r(\tau-t)+M(\tau-v)-M(t-v)]} a(v,\tau) = 0. \tag{A6}$$

Equation (1b) implies that the household's lifetime budget constraint is

$$\begin{aligned}
&\int_{\tau}^{\infty} e^{-[r(s-\tau)+M(s-v)-M(\tau-v)]} c(v,s)ds = \\
&-\int_{\tau}^{\infty} e^{-[r(s-\tau)+M(s-v)-M(\tau-v)]} \{\dot{a}(v,s) - [r(s)+m(s-v)]a(v,s)\}ds \\
&+\int_{s=\tau}^{\infty} w(s) e^{-[r(s-\tau)+M(s-v)-M(\tau-v)]} ds.
\end{aligned} \tag{A7}$$

Substituting equation (2a) into equation (A7), we have:

$$
\begin{aligned}
&\int_{\tau}^{\infty} e^{-[\rho(s-\tau)+M(s-v)-M(\tau-v)]} c(v,\tau) ds = \\
&-\int_{\tau}^{\infty} e^{-[r(s-\tau)+M(s-v)-M(\tau-v)]} \{\dot{a}(v,s)-[r(s)+m(s-v)]a(v,s)\} ds \\
&+\int_{s=\tau}^{\infty} w(s) e^{-[r(s-\tau)+M(s-v)-M(\tau-v)]} ds.
\end{aligned}
\tag{A8}
$$

According to the basic definition of integral, we have:

$$
\begin{aligned}
&-\frac{c(v,\tau)}{m(s-v)+\rho} e^{-[\rho(s-\tau)+M(s-v)-M(\tau-v)]}\Big|_{s=\tau}^{s=\infty} = \\
&-a(v,s)e^{-[r(s-\tau)+M(s-v)-M(\tau-v)]}\Big|_{s=\tau}^{s=\infty} + \int_{s=\tau}^{\infty} w(s) e^{-[r(s-\tau)+M(s-v)-M(t-v)]} ds,
\end{aligned}
\tag{A9}
$$

$$
\frac{c(v,\tau)}{m(\tau-v)+\rho} = a(v,\tau) + \int_{s=\tau}^{\infty} w(s) e^{-[r(s-\tau)+M(s-v)-M(\tau-v)]} ds. \tag{A10}
$$

Define the human wealth as:

$$
h(v,\tau) = \int_{s=\tau}^{\infty} w(s) e^{-[r(s-\tau)+M(s-v)-M(t-v)]} ds. \tag{A11}
$$

Equation (A10) determines consumption as a function of total wealth:

$$
c(v,\tau) = [m(\tau-v)+\rho][a(v,\tau)+h(v,\tau)]. \tag{A12}
$$

The simplification resulting from logarithmic utility is that the marginal propensity to consume out of total wealth is given by $[m(\tau - v) + \rho]$. Equation (A12) represents the individual consumption function, as expressed in equation (2b).

Appendix 3

This appendix derives the equilibrium patent value.

By substituting equations (7a), (7b), and (6c) into equation (6b), and subsequently integrating, we obtain the following result:

$$\mathrm{E}[V(t,t)] = \int_t^{t+T} \pi^m(j) e^{-\int_t^\tau r(z)dz} dt = \int_t^{t+T} \frac{(1-\alpha)(\alpha^2 B)^{\frac{1}{1-\alpha}} L e^{-r(\tau-t)}}{\alpha} d\tau$$

$$= -\frac{(1-\alpha)(\alpha^2 B)^{\frac{1}{1-\alpha}} L e^{-r(\tau-t)}}{r\alpha} \cdot \Bigg|_{\tau=t}^{\tau=t+T} = \frac{(1-\alpha)(\alpha^2 B)^{\frac{1}{1-\alpha}} L(1-e^{-rT})}{r\alpha}. \qquad \text{(A13)}$$

Equation (A13) represents the R&D patent value, as given by equation (8a).

Appendix 4

This appendix derives the consumer welfare function.

By applying integration by parts, equation (1a) can be expressed as follows:

$$u(v,t) = \int_t^\infty \ln c(v,\tau) e^{-[\rho(\tau-t)+M(\tau-v)-M(t-v)]} d\tau$$

$$= -\int_t^\infty \frac{\ln c(v,\tau)}{\rho + m(\tau - v)} de^{-[\rho(\tau-t)+M(\tau-v)-M(t-v)]}$$

$$= -\frac{\ln c(v,\tau) e^{-[\rho(\tau-t)+M(\tau-v)-M(t-v)]}}{\rho + m(\tau-v)}\Bigg|_{\tau=t}^{\tau=\infty} + \int_t^\infty e^{-[\rho(\tau-t)+M(\tau-v)-M(t-v)]} d\frac{\ln c(v,\tau)}{\rho + m(\tau-v)}$$

$$= \frac{\ln c(v,t)}{\rho + m(t-v)} + \int_t^\infty \frac{e^{-[\rho(\tau-t)+M(\tau-v)-M(t-v)}}{\rho + m(\tau-v)} \left[\frac{\dot{c}(v,\tau)}{c(v,\tau)} - \frac{\ln c(v,\tau) m'(\tau-v)}{\rho + m(\tau-v)}\right] d\tau. \qquad \text{(A14)}$$

Substituting equation (2a) into equation (A14), we have:

$$u(v,t) = \frac{\ln c(v,t)}{\rho + m(t-v)} + \int_t^\infty \frac{(r-\rho) e^{-[\rho(\tau-t)+M(\tau-v)-M(t-v)]}}{\rho + m(\tau-v)} d\tau$$

$$-\int_t^\infty \frac{\ln c(v,\tau) m'(\tau-v) e^{-[\rho(\tau-t)+M(\tau-v)-M(t-v)]}}{[\rho + m(\tau-v)]^2} d\tau$$

$$= \frac{\ln c(v,t)}{\rho + m(t-v)} - \frac{(r-\rho) e^{-[\rho(\tau-t)+M(\tau-v)-M(t-v)]}}{[\rho + m(\tau-v)]^2}\Bigg|_{\tau=t}^{\tau=\infty}$$

$$
\begin{aligned}
&-\int_t^{\infty} \frac{\ln c(v,\tau) m'(\tau-v) e^{-[\rho(\tau-t)+M(\tau-v)-M(t-v)]}}{[\rho+m(\tau-v)]^2} d\tau \\
&= \frac{\ln c(v,t)}{m(t-v)+\rho} + \frac{r(t)-\rho}{[m(t-v)+\rho]^2} - \int_t^{\infty} \frac{\ln c(v,\tau) m'(\tau-v) e^{-[\rho(\tau-t)+M(\tau-v)-M(t-v)]}}{[\rho+m(\tau-v)]^2} d\tau. \quad \text{(A15)}
\end{aligned}
$$

Equation (A15) defines the individual welfare function, which is composed of three distinct components:

1. The first term represents baseline welfare, ensuring that current consumption levels are sustained into the future, adjusted for mortality probabilities.
2. The second term captures welfare gains arising from anticipated future consumption growth, also adjusted for mortality.
3. The third term accounts for the discounted welfare loss associated with the increase in mortality rate as individual age.

Furthermore, it is worth noting that the discounted welfare loss due to rising mortality rates with age is negligible ($\ln c(v, \tau) m'(\tau - v) e^{-[\rho(\tau - t) + M(\tau - v) - M(t - v)]} \rightarrow 0$). As such, the effect of the third term is minor compared to the first two components, and it is subsequently excluded from further analysis.

According to equation (A15), we have:

$$
u(v,t) \cong \frac{\ln c(v,t)}{m(t-v)+\rho} + \frac{r(t)-\rho}{[m(t-v)+\rho]^2}, \quad \text{(A16)}
$$

$$
u(s,s) \cong \frac{\ln c(s,s)}{m(0)+\rho} + \frac{r(s)-\rho}{[m(0)+\rho]^2}. \quad \text{(A17)}
$$

We consider that the total population unit is normalized to 1.

$$
\int_{-\infty}^{t} \beta e^{-M(t-v)} dv = \left. \frac{\beta e^{-M(t-v)}}{m(t-v)} \right|_{v=-\infty}^{v=t} = \frac{\beta}{m(0)} = 1. \quad \text{(A18)}
$$

Bringing equations (A17) and (A18) into the definition of welfare for future generations yields:

$$u(s,t) = e^{-\delta(s-t)}u(s,s) \cong e^{-\delta(s-t)}\left[\frac{\ln c(s,s)}{\beta+\rho} + \frac{r(s)-\rho}{(\beta+\rho)^2}\right]. \qquad \text{(A19)}$$

Equations (A16) and (A19) correspond to the welfare function equations (11a) and (12a) for individual consumers, respectively.

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死亡率隨年齡遞增下的最適專利長度

金志婷*

摘　要

代表性個人的總體經濟模型所估計的最適專利長度比目前的實際專利長度長很多。而在固定死亡率的假設下，考慮永遠年輕的重疊代模型，延長專利會擴大跨代間的貧富差距，所估計的最適專利長度較短，但仍長於目前實際的專利長度。當我們考慮到死亡率會隨年齡的增加而增加時，可以從專利延期中受益的老年人之死亡率更高，縮短了最適專利的期限。雖然仍然比目前的實際專利要長，但已經非常接近了。

關鍵詞：社會福利、世代重疊、專利長度、研究與發展 (R&D)、跨代財富分配

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